# MODULAR FRAMIZATION OF THE BMW ALGEBRA

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## 1. Introduction

In this work we introduce the concept of Modular Framization or simply Framization. The idea originates from the Yokonuma–Hecke algebra, shortened to Y–H algebra, built from the classical Hecke algebra by framization. That is, the Y–H algebra is built by adding framing generators to the Hecke algebra and changing the Hecke algebra quadratic relation by a new quadratic relation which involves the framing generators. Using then the Y–H algebras and a Markov trace constructed on them[4] we can produce invariants for oriented framed knots[5, 7], classical knots[8] (which satisfies a cubic skein relation), and singular knots[6]. Moreover, we have connections of the Y–H algebras with virtual knots and transversal knot theory.

All these results invite us to try and apply the framization mechanism on other knot algebras, that is, algebras having generators whose behavior involves braid relations and polynomial relations. For example, the Temperley–Lieb algebra, the BMW algebra, the singular Hecke algebra introduced by Paris and Rabenda, the Hecke algebra of $B$ type, et cetera.

Let $d$ and $n$ be two positive integers. The aim of this note is to construct a framization $F_{d,n}$ of the Birman–Wenzl–Murakami algebra, also known as BMW algebra, and start a systematic study of this framization. We show that $F_{d,n}$ is finite dimensional and the 'braid generators' of this algebra satisfy a quartic relation which is of minimal degree not containing the generators $t_i$. They also satisfy a quintic relation, as the smallest closed relation. We conjecture that the algebras $F_{d,n}$ support a Markov trace which allow to define polynomial invariants for unoriented knots in an analogous way that the Kauffman polynomial is derived from the BMW algebra.

in Section 3, we give some more knot algebras on which the framization can be applied, such as the Temperley–Lieb algebra, the singular Hecke algebra and Hecke algebras of type $B$.

## 2. The modular framization of BMW

Let $K$ denote the field of rational functions $\mathbb{C}(l, m)$, where $l$ and $m$ are two unspecified parameters. For any natural number $n$, J. Birman and H. Wenzl [1] and, simultaneously but independently, J. Murakami [13] defined a unital associative $K$–algebra of two parameters, $C_n = C_n(l, m)$, which is known as the Birman–Wenzl–Murakami algebra or, simply,

1991 *Mathematics Subject Classification.* 57M27, 20C08, 20F36.

*Key words and phrases.* modular framization, BMW algebra, Yokonuma–Hecke algebra, Temperley–Lieb algebra, singular Hecke algebra, $B$–type Hecke algebras.

Both authors were partially supported by Fondecyt 1085002, NTUA and Dipuv.

the BMW algebra. The algebra $C_n$ with unity 1 is defined by two sets of generators, $g_1, \ldots, g_{n-1}$ and $h_1, \ldots, h_{n-1}$, satisfying: the *braid relations* among the $g_i$'s:

(B1) $\quad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad$ and $\quad$ (B2) $\quad g_i g_j = g_j g_i \quad$ for $\quad |i-j| > 1$

together with the following relations:

$$g_i^2 = 1 - m g_i + m l^{-1} h_i \tag{1}$$

$$g_i h_i = l^{-1} h_i \tag{2}$$

$$h_i g_{i\pm 1} h_i = l h_i \tag{3}$$

From the defining relations of $C_n$ we deduce that the $g_i$'s are invertible and also the following important relations:

$$g_i^{-1} = g_i - m h_i + m \tag{4}$$

$$h_i g_i = l^{-1} h_i \tag{5}$$

$$h_i h_j = h_j h_i, \quad \text{for } |i-j| \geq 2 \tag{6}$$

$$h_i^2 = y h_i \tag{7}$$

where

$$y := 1 + \frac{l^{-1} - l}{m}$$

Other useful relations, which can be deduced from the above relations (see [15]), are:

$$h_i h_{i\pm 1} h_i = h_i \tag{8}$$

$$g_{i\pm 1} g_i h_{i\pm 1} = h_i g_{i\pm 1} g_i = h_i h_{i\pm 1} \tag{9}$$

$$g_{i\pm 1} h_i g_{i\pm 1} = g_i^{-1} h_{i\pm 1} g_i^{-1} \tag{10}$$

$$g_{i\pm 1} h_i h_{i\pm 1} = g_i^{-1} h_{i\pm 1} \tag{11}$$

$$h_{i\pm 1} h_i g_{i\pm 1} = h_{i\pm 1} g_i^{-1} \tag{12}$$

**Definition 1.** Let $d$ be a natural number and let $y_0 := y$ and $y_1, \ldots, y_{d-1}$ be unspecified parameters. The *$d$–framization of the algebra $C_n$*, denoted $F_{d,n} = F_{d,n}(l, m, y_0, \ldots, y_{d-1})$, is defined as follows. The algebra $F_{d,n}$ is the unital (with unity 1) associative algebra over $K$, defined through three sets of generators: the two sets of generators of the algebra $C_n$ given above, together with 'framing generators' generators $t_1, \ldots, t_n$, satisfying all defining relations of $C_n$, except the quadratic relation in Eq. 1, that is, (B1), (B2), (2), (3), together with the following relations:

$$t_i^d = 1, \quad t_i t_j = t_j t_i \qquad \text{for all } i, j \tag{13}$$

$$t_i h_i = t_{i+1} h_i, \quad h_i t_i = h_i t_{i+1} \tag{14}$$

$$h_i t_i^k h_i = y_k h_i \quad 0 \leq k \leq d-1 \tag{15}$$

$$t_j g_i = g_i t_{s_i(j)} \qquad \text{for all } i, j \tag{16}$$

where $s_i(j)$ is the effect of the transposition $s_i = (i, i+1)$ on $j$, and the quadratic relation

$$g_i^2 = (1-m) - m e_i (g_i - 1) + m l^{-1} h_i \tag{17}$$

where

$$e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s} \tag{18}$$

**Remark 1.** In the case $d = 1$ we have $e_i = 1$, hence $F_{d,n}$ coincides with $C_n$.

**Remark 2.** Mapping $t_i \mapsto 1$, $g_i \mapsto g_i$ and $h_i \mapsto h_i$ defines an algebra epimorphism from the algebra $F_{d,n}$ onto the algebra $C_n$.

**Proposition 1.** *For all $i$ we have:*

(1) $e_i^2 = e_i$
(2) $e_i h_i = h_i e_i = h_i$
(3) *The elements $g_i$ are invertible and*

$$g_i^{-1} = \frac{1}{1-m} g_i - \frac{m}{1-m} g_i e_i - m h_i + m e_i$$

*Proof.* Claim (1) is easy to prove, see [5]. Claim (2) comes from Eqs. 14. Finally, using (1) and (2) of the proposition and Eq. 17, Eq. 2 and Eq. 5, one can verify that $g_i^{-1} g_i = g_i g_i^{-1} = 1$. Thus (3) is also proved. □

**Proposition 2.** *The elements $g_i$ satisfy the quartic relation*

$$g_i^4 + m g_i^3 + (m-2) g_i^2 + m(m-1) g_i - (m-1) = m l^{-1} \left( m + l^{-2} - 1 \right) h_i \tag{19}$$

*and this is of minimal degree not containing the generators $t_i$. Also, they satisfy the 'closed'quintic equation,*

$$\left( x - l^{-1} \right) \left( x^4 + m x^3 + (m-2) x^2 + m(m-1) x - (m-1) \right) = 0 \tag{20}$$

*and this is of minimal degree not containing the generators $t_i$ and $h_i$. Notice that* $x^4 + m x^3 + (m-2) x^2 + m(m-1) x - (m-1) = (x^2 + m x - 1)(x^2 + m - 1)$.

*Proof.* Multiplying Eq. 17 by $e_i$ and solving with respect to $m e_i (g_i - 1)$, we obtain

$$m e_i (g_i - 1) = (1-m) e_i - e_i g_i^2 + m l^{-1} h_i \tag{21}$$

Also, solving $m e_i (g_i - 1)$ directly from Eq. 17 we have $m e_i (g_i - 1) = (1-m) - g_i^2 + m l^{-1} h_i$. Hence

$$e_i (g_i^2 + m - 1) = (g_i^2 + m - 1) \tag{22}$$

Multiplying now Eq. 17 by $g_i$ we have $g_i^3 = (1-m) g_i - m e_i g_i^2 + m e_i g_i + m l^{-2} h_i$. Then

$$g_i^3 (1-m) g_i - m e_i (g_i^2 + m - 1) + m^2 e_i + m e_i (g_i - 1) + m l^{-2} h_i$$

so, from Eq. 22 we obtain

$$g_i^3 = (1-m) g_i - m (g_i^2 + m - 1) + m^2 e_i + m e_i (g_i - 1) + m l^{-2} h_i$$

Replacing in this last equation the expression in Eq. 21 for $me_i(g_i - 1)$, we obtain

$$g_i^3 = -(m+1)g_i^2 + (1-m)g_i + m(l^{-1} + l^{-2})h_i + (1-m^2) + m^2 e_i$$

Hence,

$$me_i = \frac{1}{m}\left[g_i^3 + (m+1)g_i^2 + (m-1)g_i - m(l^{-1} + l^{-2})h_i + (m^2-1)\right]$$

Then, replacing this last expression for $me_i$ in Eq. 17, we obtain Eq. 19. Finally, Eq. 5 says $h_i(g_i - l^{-1}) = 0$, and substituting $h_i$ by Eq. 19 we obtain Eq. 20. $\square$

2.1. *Topological interpretations.* Note that the generators $t_1, \ldots, t_n$ together with the relations in Eq. 13 form a copy of the abelian group $(\mathbb{Z}/d\mathbb{Z})^n$ and recall[5, 6, 7, 8] that the modular framed braid group $\mathcal{F}_{d,n}$ is given as $\mathcal{F}_{d,n} = (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$, where the action of $B_n$ on $(\mathbb{Z}/d\mathbb{Z})^n$ is given by the permutation induced by a braid on the indices. Geometrically, elements of $\mathcal{F}_{d,n}$ are classical braids on $n$ strands with an integer modulo $d$, the framing, attached to each strand. In particular, the framing generator $t_i$ stands for the identity framed braid with framing 1 on the $i$th strand and 0 elsewhere. Looking at our baby-example, the Y-H algebra, it is shown in [5] that it can be defined as a quotient of a group algebra of $\mathcal{F}_{d,n}$ over an ideal generated by the quadratic relations in the Y-H algebra.

Passing now to the BMW algebra, it is a quotient of the classical braid group $B_n$. To see this consider the algebra $C_n$ generated by the $g_i$'s only, and view Eq. 4 as the defining relations for the $h_i$'s. Analogously, its $d$–framization $F_{d,n}$ contains a copy of the abelian group $(\mathbb{Z}/d\mathbb{Z})^n$, generated by $t_1, \ldots, t_n$. So, if we exempt the $h_i$'s from the set of generators for the algebra $F_{d,n}$ and if we consider (3) of Proposition 1 as the defining relation for the $h_i$'s, we conclude that $F_{d,n}$ can be seen as a quotient of an appropriate group algebra of $\mathcal{F}_{d,n}$. Further, the element $h_i$ can be seen represented in the category of $(n,n)$–tangles as the elementary tangle consisting in two curved parallel horizontal arcs joining the endpoints $i$ and $i+1$ at the top and at the bottom of the otherwise identity tangle. So, elements in the algebra $F_{d,n}$ can be viewed as framed $(n,n)$–tangles, with framings modulo $d$.

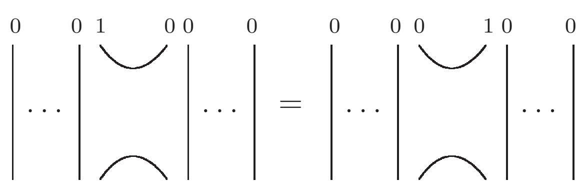


FIGURE 1. The relation $t_i h_i = t_{i+1} h_i$

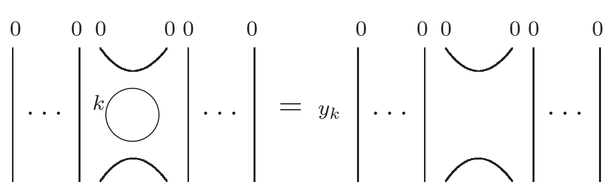


FIGURE 2. The relation $h_i t_i^k h_i = y_k h_i$

The elements $e_i$ seen as elements of $\mathbb{C}\mathcal{F}_{d,n}$ can be interpreted geometrically as the average of the sum of $d$ identity framed braids with framings as shown below for $e_1$.

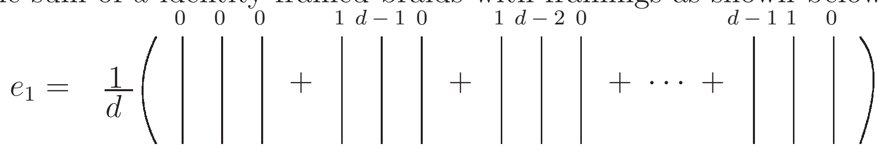


FIGURE 3. The element $e_1 \in \mathbb{C}\mathcal{F}_{d,3}$

2.2. *$F_{d,n}$ is finite dimensional.*

**Proposition 3.** *Any element in $F_{d,n}$ can be written as a $K$–linear combination of monomials of the form $\alpha f \beta$, where $\alpha$ and $\beta$ are monomials in $1, g_1, \ldots, g_{n-2}$, $h_1, \ldots, h_{n-2}$, $t_1, \ldots, t_{n-1}$ and $f \in X_n := \{t_n^s, g_{n-1}, t_{n-1}^s h_{n-1} t_{n-1}^r\,;\, 1 \leq r, s \leq d\}$.*

*Proof.* The proof is by induction on $n$. For $n = 2$ the proposition follows directly from the defining relations of $F_{d,2}$, that is, from Eq. 2, Eq. 3 and Eqs. 13–17. We assume now truth of the proposition for $2 < k \leq n-1$. An arbitrary element in $F_{d,n}$ is a $K$–linear combination of monomials $M$ in $1, g_1, \ldots, g_{n-1}$, $h_1, \ldots, h_{n-1}$, $t_1, \ldots, t_n$. We shall check first the case where $M$ is such a monomial containing two elements $f_1, f_2$ of $X_n$. Further, by the induction hypothesis, we may assume that $M$ is in the form

$$M = M_1 f_1 M_2 f_2 M_3$$

where $M_i$ are monomials in $1, g_1, \ldots, g_{n-2}$, $h_1, \ldots, h_{n-2}$, $t_1, \ldots, t_{n-1}$, each with at most one $f \in X_{n-1}$. We have two cases, according to whether $M_2$ contains or not an element in $X_{n-1}$. In the case where $M_2$ does not have such an element, we have:

$$M = M_1 M_2 f_1 f_2 M_3$$

So, we must reduce $f_1 f_2$ as a linear combination of monomials conforming with the statement of the proposition. We have nine cases to reduce, but we shall consider the more representative ones, that is the cases: $f_1 = g_{n-1}$ and $f_2 = t_{n-1}^s h_{n-1} t_{n-1}^r$. Applying Eqs. (16), (5) and (14), respectively, we obtain that $f_1 f_2$ can be reduced:

$$f_1 f_2 = t_n^s g_{n-1} h_{n-1} t_{n-1}^r = l^{-1} t_n^s h_{n-1} t_{n-1}^r = l^{-1} t_{n-1}^s h_{n-1} t_{n-1}^r$$

Suppose now that $M_2$ contains one element $1 \neq f \in X_{n-1}$. So, we can assume that $M$ is of the form $M = M' f_1 f f_2 M''$, where $M', M'' \in F_{d,n-1}$ and $f_1, f_2 \in X_n$. Hence, it is enough to show that $f_1 f f_2$ is a linear combinations of monomials as in the statement of the proposition; we shall show the reduction for only one case, as the rest of the 26 cases follow in the same way. Suppose $f_1 = f_2 = g_{n-1}$ and $f = t_{n-2}^s h_{n-2} t_{n-2}^r$. We have:

$$\begin{aligned}
f_1 f f_2 = g_{n-1} t_{n-2}^s h_{n-2} t_{n-2}^r g_{n-1} &= t_{n-2}^s g_{n-1} h_{n-2} g_{n-2} t_{n-1}^r && \text{(from Eq. 16)}\\
&= t_{n-2}^s g_{n-2}^{-1} h_{n-1} g_{n-2}^{-1} t_{n-2}^r && \text{(from Eq. 10)}\\
&= g_{n-2}^{-1} t_{n-1}^s h_{n-1} t_{n-1}^r g_{n-2}^{-1} && \text{(from Eq. 16)}
\end{aligned}$$

Using the expression of $g_{n-2}^{-1}$ in Proposition 1, the reduction of $f_1 f f_2$ follows. □

**Corollary 1.** *$F_{d,n}$ is finite dimensional.*

## 3. Speculations

Apart from the framization of the BMW algebra, defined in Section 2, we give below some more algebras on which the framization can be applied, recalling first the Yokonuma–Hecke algebra, which is the framization of the classical Hecke algebra of type $A$.

*The modular framization of the Iwahori–Hecke algebra.* This is the Yokonuma–Hecke algebra, shortened to Y–H algebra. We recall the definition of the Y–H algebras. Fix $u \in \mathbb{C}\backslash\{0,1\}$. Given two positive integers $d$ and $n$, we denote $\mathrm{Y}_{d,n} = \mathrm{Y}_{d,n}(u)$ the Yokonuma–Hecke algebra, which is a unital associative algebra over $\mathbb{C}$, defined by the generators $1, g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$, satisfying: the braid relations (B1), (B2) for the $g_i$'s, relations (13), (16) for the $t_i$'s, together with the extra *Yokonuma quadratic relations*:

$$g_i^2 = 1 + (u-1)e_i(1-g_i) \tag{23}$$

where $e_i$ as given in Eq. 18. In the case $d = 1$, $\mathrm{Y}_{d,n}$ coincides with the classical Iwahori–Hecke algebra of type $A$, $H_n(q)$. That is, the Y–H algebra is built by adding framing generators to $H_n(q)$ and changing the *Hecke quadratic relations*:

$$g_i^2 = q \cdot 1 + (q-1)g_i \tag{24}$$

by the quadratic relations Eq. 23, which involve the framing generators. Alternatively, as shown in [5], the Y–H algebra can be defined as a quotient of a group algebra of the modular framed braid group $\mathcal{F}_{d,n}$ over an ideal generated by the quadratic relations Eq. 23. We note that, if we considered the quotient of a group algebra of the classical or the modular framed braid group over the quadratic relations Eq. 24 we would obtain nothing more interesting, comparing to the much richer structure of the Y–H algebra.

Using the Y–H algebras and a Markov trace constructed on them[4] we have constructed invariants for oriented framed knots[5, 7], classical knots[8] and singular knots[6]. In the Y–H algebra the following closed cubic relation is satisfied, not involving the framing generators, which gives rise to a cubic skein relation for the invariant of classical knots.

$$g_i^3 = -ug_i^2 + g_i + u \tag{25}$$

*The modular framization of the Temperley–Lieb algebra.* In [9] we define the framization of the classical Temperley–Lieb algebra, $\mathrm{TL}_{d,n}(u)$, the Yokonuma–Temperley–Lieb algebra, $\mathrm{YTL}_{d,n}(u)$, as follows.

**Definition 2.** The Yokonuma–Temperley–Lieb algebra, $\mathrm{YTL}_{d,n}(u)$, is defined as the following quotient of the Yokonuma-Hecke algebra:

$$\mathrm{YTL}_{d,n}(u) = \frac{\mathrm{Y}_{d,n}(u)}{\langle g_ig_jg_i + g_ig_j + g_jg_i + g_i + g_j + 1, \quad |i-j| = 1\rangle}$$

In [9] we also find appropriate inductive bases for the Yokonuma–Temperley–Lieb algebras and we construct a Markov trace on them. Using this trace we define topological invariants for various types of knots, in analogy to the case of the Y–H algebras.

*The modular framization of the singular Hecke algebra.* A definition of the singular Hecke algebra, denoted $SH_n(q)$, was proposed by Paris and Rabenda [14]. This algebra is defined as the quotient of the group algebra of the singular braid monoid $SB_n$ over the Hecke algebra quadratic relations Eq. 24. Recall that the monoid $SB_n$ is defined by: the unit 1, the classical elementary braids $\sigma_i$ with their inverses $\sigma_i^{-1}$, $1 \leq i \leq n-1$, which are subject to the braid relations (B1), (B2), and by the elementary singular braids $\tau_i$, $1 \leq i \leq n-1$, together with the following relations:

$$\begin{aligned} [\sigma_i, \tau_j] = [\tau_i, \tau_j] &= 0 \quad \text{for} \quad |i-j| > 1 \\ [\sigma_i, \tau_i] &= 0 \quad \text{for all} \quad i \\ \sigma_i \sigma_j \tau_i &= \tau_j \sigma_i \sigma_j \quad \text{for} \quad |i-j| = 1 \end{aligned} \tag{26}$$

Keeping the same notation for $\tau_i$ in the quotient $SH_n(q)$ and corresponding $g_i$ to $\sigma_i$, the singular Hecke algebra $SH_n(q)$ is the complex associative unital algebra defined by the generators $1, g_1 \ldots g_{n-1}$, $\tau_1, \ldots \tau_{n-1}$ with the relations Eq. 24 together with the relations in Eq. 26, placing $g_i$ instead of $\sigma_i$.

**Definition 3.** Let $d$ be a natural number. The *$d$–framization of the algebra $HS_n(u)$*, denoted $FS_{d,n} = FS_{d,n}(u)$, is defined as follows. The algebra $FS_{d,n}$ is the unital (with unity 1) associative algebra over $\mathbb{C}$, defined through three sets of generators: the two sets of generators of the algebra $SH_n(q)$ given above, together with generators $t_1, \ldots, t_n$, satisfying all defining relations of $SH_n(u)$, except the quadratic relation in Eq. 24, together with the relations of Eqs. 13, (16) and the Yokonuma quadratic relations in Eqs. 23.

*The modular framization of $B$–type Hecke algebras.* Recall that the Artin braid group of type $B$, which we denote by $B_{1,n}$, is defined by generators $T, \sigma_1, \ldots, \sigma_{n-1}$, satisfying the braid relations (B1), (B2) for the $\sigma_i$'s and the following *$B$–type relations*:

$$\sigma_1 T \sigma_1 T = T \sigma_1 T \sigma_1, \quad \sigma_i T = T \sigma_i \qquad \text{if} \quad i > 1 \tag{27}$$

For $q, Q \in \mathbb{C}\backslash\{0,1\}$, the classical Iwahori–Hecke algebra of type $B$, $H_n(q,Q)$, can be seen as a quotient of the group algebra $\mathbb{C}B_{1,n}$ by factoring out the ideal generated by Eqs. 24, where $g_i$ denotes the image of $\sigma_i$ in $H_n(q,Q)$, and the relation:

$$T^2 = (Q-1)T + Q \tag{28}$$

Further, for $q, u_1, \ldots, u_r \in \mathbb{C}\backslash\{0,1\}$, the cyclotomic Hecke algebra of type $B$ and of degree $r$, $H_n(q,r)$, can be defined as the quotient of the group algebra $\mathbb{C}B_{1,n}$ by factoring out the ideal generated by the relations Eqs. 24 and the *cyclotomic relation*:

$$(T-u_1)(T-u_2)\cdots(T-u_r) = 0 \tag{29}$$

Finally, the generalized Hecke algebra of type $B$, $H_n(q,\infty)$, is defined[12] as the quotient of the group algebra $\mathbb{C}B_{1,n}$ over the relations Eqs. 24 only. In [11, 12] Markov traces are constructed on all these algebras, giving rise to Jones–type invariants of knots in the solid torus.

**Definition 4.** The *framed braid group of type $B$*, $\mathcal{F}_{1,n}$, is defined as $\mathcal{F}_{1,n} = \mathbb{Z}^n \rtimes B_{1,n}$, where the action of $B_{1,n}$ on $\mathbb{Z}^n$ is given by the permutation induced by a braid of type $B$ on the indices. Geometrically, elements of $\mathcal{F}_{1,n}$ are braids on $n+1$ strands with the first strand fixed and with an integer, the framing, attached to each one of the rest $n$ strands.

The *modular framed braid group of type B*, $\mathcal{F}_{d,1,n}$, is defined similarly, except that $B_{1,n}$ acts on $(\mathbb{Z}/d\mathbb{Z})^n$ and the framings are modulo $d$. So, the group $\mathcal{F}_{d,1,n}$ is defined by generators $T, \sigma_1, \ldots, \sigma_{n-1}, t_1, \ldots, t_n$, satisfying the braid relations (B1), (B2) for the $\sigma_i$'s, relations Eqs. 27 for the generator $T$ and relations (13), (16) for the framing generators $t_i$'s.

**Definition 5.** Let $d$ be a natural number. The *d–framization of the Iwahori–Hecke algebra of B–type* $H_n(q,Q)$, denoted $FH_{d,Q,n} = FH_{d,Q,n}(u)$, is defined as the quotient of the modular framed braid group of type $B$, $\mathcal{F}_{d,1,n}$, over the quadratic relations in Eq. 28 and Eqs. 23. The *d–framization of the cyclotomic Hecke algebra* $H_n(q,r)$, denoted $FH_{d,r,n} = FH_{d,r,n}(u)$, is defined as the quotient of the modular framed braid group of type $B$, $\mathcal{F}_{d,1,n}$, over the quadratic relations in Eq. 29 and Eqs. 23. Finally, the *d–framization of the generalized Hecke algebra of type B,* $\mathcal{H}_n(q,\infty)$, denoted $FH_{d,n} = FH_{d,n}(u)$, is defined as the quotient of the modular framed braid group of type $B$, $\mathcal{F}_{d,1,n}$, over the quadratic relations in Eqs. 23.

$B$–type framizations are studied in [10], including the $B$–type Temperley–Lieb algebras and $B$–type BMW algebras.

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